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MATHEMATICS

TARGET IIT JEE

FUNCTION

THEORY AND EXERCISE BOOKLET

CONTENTS

JEE Syllabus :

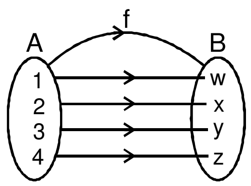
Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions,

A. DEFINITION

Function is defined as a rule or a manner or a mapping or a correspondence f which maps each & every element of set A with a unique element of set B. It is denoted by :

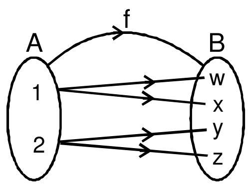
$f : A \rightarrow B$ or $A \xrightarrow{f} B$ we read it as " f is a function from A to B"

Ex. f maps A to B



Yes

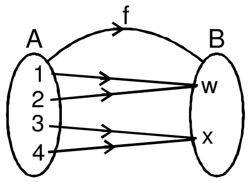
Figure (1)



No

Figure (2)

Figure –2 does not represent a function because conversion is allowed (figure–3) But diversion is not allowed.



Yes

Figure (3)

Ex.1 Which of the following correspondences can be called a function ?

(A) $f(x) = x^3$; $\{-1, 0, 1\} \rightarrow \{0, 1, 2, 3\}$ (B) $f(x) = \pm\sqrt{x}$; $\{0, 1, 2\} \rightarrow \{-2, -1, 0, 1, 2\}$

(C) $f(x) = \sqrt{x}$; $\{0, 1, 4\} \rightarrow \{-2, -1, 0, 1, 2\}$ (D) $f(x) = -\sqrt{x}$; $\{0, 1, 4\} \rightarrow \{-2, -1, 0, 1, 2\}$

Sol. f(x) in (C) & (D) are functions as definition of function is satisfied. while in case of (A) the given relation is not a function, as $f(-1) \notin$ codomain. Hence definition of function is not satisfied. While in case of (B), the given relation is not a function, as $f(1) = \pm 1$ and $f(4) = \pm 2$ i.e. element 1 as well as 4 in domain are related with two elements of codomain. Hence definition of function is not satisfied.

Ex.2 If X = {a, b, c, d, e} & Y = {p, q, r, s, t} then which of the following subset(s) of X × Y is/are a function from X to Y.

(A) {(a, r) (b, r) (b, s) (d, t) (e, q) (c, q)} (B) {(a, r) (b, p) (c, t) (d, q)}

(C) {(a, p) (b, t) (c, r) (d, s) (e, q)} (D) {(a, r) (b, r) (c, r) (d, r) (e, r)}

Sol. Let us check every option for the two conditions of the function

(A) $\because$ b has two output (images) namely r & s $\therefore$ Not a function

(B) $\because$ $e \in X$ does not have any image $\therefore$ Not a function

(C) $\because$ every element of X has one and only one output $\therefore$ it is a function

(D) $\because$ every element's output is r $\therefore$ it is a function

Hence correct options are (C) & (D).

"Function" as an ordered pair :

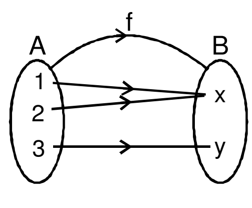
$f : A \rightarrow B$

f : {(1, x), (2, x), (3, x)}

$\Rightarrow f \subset A \times B$

where A × B is the cartesian product of two set A & B

= {(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)}



Remark : Every function from A to B satisfied the following Relation :

(1) $f \subset A \times B$

(2) Notation : $\forall\ a \in A\ \exists\ b \in B\ |\ b = f(a)$

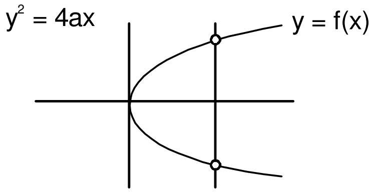
$\forall$ (For all) $\exists$ (there exist) $|$ (such that)

(3) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(4) In a graphical representation of a function $y = f(x)$.

If vertical line cuts the curve more than once then it is not a function. It is called as vertical line test

Ex :



It is not a function as vertical line touches curve at more than once point.

B. DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION

Let $f : A \to B$, then the set A is known as the domain of f& the set B is known as co-domain of f. If a member 'a' of A is associated to the member 'b' of B, then'b' is called the **f-image** of 'a' and we write $b = f(a)$. Further 'a' is called a **pre-image** of 'b'. The set $\{f(a) : \forall\ a \in A\}$ is called the **range** of f and is denoted by f(A). Clearly $f(A) \subseteq B$.

If only expression of f(x) is given (domain and codomain are not mentioned), then domain is set of those values of 'x' for which f (x) is real, while codomain is considered to be $(-\infty, \infty)$ (except in ITFs)

A function whose domain and range are both subsets of real numbers is called a **real function.**

(Algebraic Operations on Functions) : If f & g are real valued functions of x with domains A and B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f . g)$ & (f/g) as follows:

(i) $(f \pm g)(x) = f(x) \pm g(x)$

(ii) $(f . g)(x) = f(x) . g(x)$

domain in each case is $A \cap B$

(iii) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B \text{ and } g(x) \neq 0\}$.

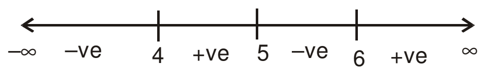
Ex.3 Find the domain of definition of the function $y = \log_{10} \dfrac{x-5}{x^2 - 10x + 24} - \sqrt[3]{x+5}$

Sol. For y to be defined

(i) $\dfrac{x-5}{x^2 - 10x + 24} > 0$

When $x - 5$, $x = 5$ and when $x^2 - 10x + 24$, $x = 4, 6$

sign scheme for $\dfrac{x-5}{x^2 - 10x + 24}$ is as follows.



Put x = 0 $\quad \therefore \dfrac{x-5}{x^2 - 10x + 24} > 0 \Rightarrow 4 < x < 5$ or $x > 6$ ...(A)

(ii) $(x+5)^{\frac{1}{3}}$ is defined for all x ...(B)

Combining (A) and (B), we get $6 < x < \infty$ or $4 < x < 5$ $\quad \therefore$ Domain = $]4, 5[\ \cup\ ]6, \infty[$

Ex.4 Find the domain of the function $f(x) = \sqrt{x^2 - x - 6} + \sqrt{6 - x}$

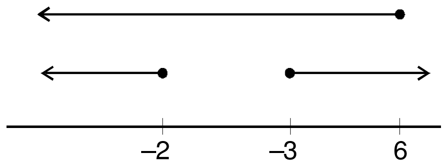
Sol. $x^2 - x - 6 \geq 0$ and $6 - x \geq 0$

$(x - 3)(x + 2) \geq 0$ $\quad$ $x \leq 6$

$x \geq 3$ or $x \leq -2$ $\quad$ $x \in (-\infty, 6]$

$x \in (-\infty, -2] \cup [3, \infty)$

$\therefore\ x \in (-\infty, -2] \cup [3, 6]$



Ex.5 Find the domain of definition of the function,

(a) $f(x) = \sqrt{(x^2 - 3x - 10)\,.\,\ell n^2\,(x - 3)}$ $\qquad$ **(b)** $\sqrt{\dfrac{1 - 5^x}{7^{-x} - 7}}$

Sol. **(a)** $(x - 5)(x + 2)\ \ell n^2\ (x - 3) \geq 0$

Equality gives $x = 5$; $x = 4$ (Note for $x = -2$ $\ell n\,(x - 3)$ is meaningless)

Now $(x - 5)(x + 2) > 0$ and $x - 3 > 0 \Rightarrow x > 5$ hence the result

Ans. : $\{4\} \cup [5, \infty)$

(b) $7^{-x} \neq 7 \Rightarrow x \neq -1$

Now $\dfrac{1 - 5^x}{7^{-x} - 7} = 0 \Rightarrow x = 0$; $\qquad$ Now solve $\dfrac{1 - 5^x}{7^{-x} - 7} > 0$

$1 - 5^x > 0$ and $7^{-x} - 7 > 0$ $\qquad$ or $1 - 5^x < 0$ and $7^{-x} - 7 < 0$

Ans. : $(-\infty, -1) \cup [0, \infty)$

Ex.6 Find the Domain of the function $f(x) = \sqrt{\dfrac{x - 7}{3 - x}} - \dfrac{1}{\sqrt[3]{x^2 - 6x + 8}}$

Sol. $\dfrac{x - 7}{3 - x} \geq 0 \Rightarrow \dfrac{x - 7}{x - 3} \leq 0 \Rightarrow x \in (3, 7]$

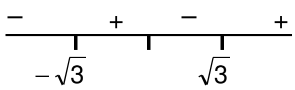
& $x^2 - 6x + 8 \neq 0 \Rightarrow (x - 4)(x - 2) \neq 0 \Rightarrow x \neq 2, 4$

Domain $x \in (3, 7] - \{4\}$

Ex.7 Find the domain of given function $f(x) = \sqrt{3x - x^3}$

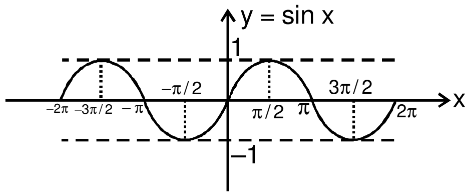
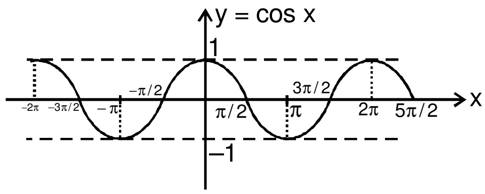
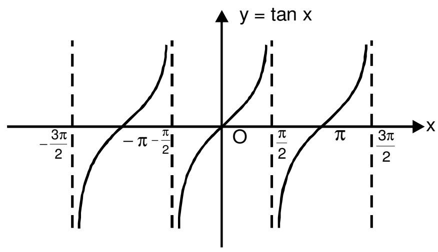
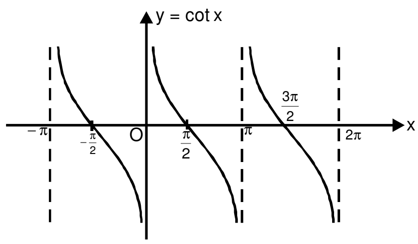
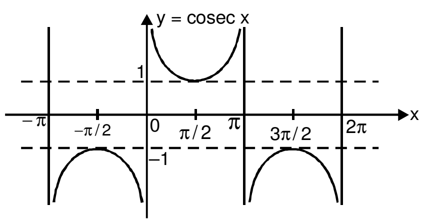
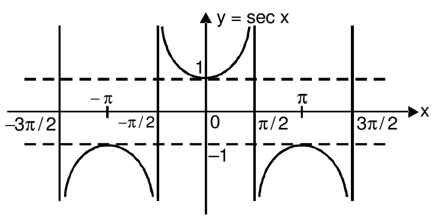
Sol. $3x - x^3 \geq 0 \Rightarrow x^3 - 3x \leq 0 \Rightarrow x(x^2 - 3) \leq 0 \Rightarrow x(x - \sqrt{3})(x + \sqrt{3}) \leq 0$

$\therefore\ x \in (-\infty, -\sqrt{3}\,] \cup [0, \sqrt{3}\,]$



C. IMPORTANT TYPE OF FUNCTIONS

(1) Trigonometric function :

Function	Domain	Range	Curve
(i) $f(x) = \sin x$	$x \in R$	$y \in [-1, 1]$	
(ii) $f(x) = \cos x$	$x \in R$	$y \in [-1, 1]$	
(iii) $f(x) = \tan x$	$x \in R - (2n + 1)\dfrac{\pi}{2}, n \in I$	$y \in R$	
(iv) $f(x) = \cot x$	$x \in R - n\pi, n \in I$	$y \in R$	
(v) $f(x) = \text{cosec } x$	$x \in R - n\pi, n \in I$	$y \in (-\infty, -1] \cup [1, \infty)$	
(vi) $f(x) = \sec x$	$x \in R - (2n+1)\dfrac{\pi}{2}, n \in I$	$y \in (-\infty, -1] \cup [1, \infty)$	

(2) Polynomial Function :

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n$

where $a_0, a_1, a_2 \ldots\ldots\ldots a_n \in R$ $\quad n \in W$

If $a_0 \neq 0$, then f(x) is called n^{th} degree polynomial and Domain $x \in R$

(3) Algebraic Function : A function is called an algebraic function. If it can be constructed using algebraic operations such as additions, subtractions, multiplication, division taking roots etc. All polynomial functions are algebraic but converse is not true.

Ex : $f(x) = \sqrt{x^4 + 5x^2} + x + (x^3 + 5)^{3/5}$, $f(x) = x^3 + 3x^2 + x + 5$

Remark : Function which are not algebraic are called as **TRANSCENDENTAL FUNCTION**.

Ex : $f(x) = \dfrac{(x^5 + 5x^2)^{3/5}}{x^3} + 3\sqrt{x^2 + 5x + 6} + \ell n\, x \rightarrow$ transcidental function

Ex : $f(x) = \sqrt{x^2 + 7} + e^{\ell n\, x} + \dfrac{x + 7}{\sqrt{x^2 + 7}} \rightarrow$ algebraic function.

(4) Rational Function : It is a function of form $f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are poly. function and $h(x) \neq 0$ **Ex.** $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 4}$

(5) Logarithmic function : $f(x) = \log_a x$, where $x > 0$, $a > 0$, $a \neq 1$

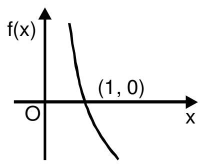
$a \rightarrow$ base, $x \rightarrow$ number or argument of log.

Case–I : $0 < a < 1$

$f(x) = \log_a x$

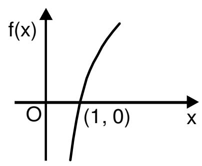
Domain : $x \in (0, \infty)$

Range : $y \in R$



Case–II : $a > 1$

$f(x) = \ell n\, x$



(6) Exponential function : $f(x) = a^x$, where $a > 0$, $a \neq 1$

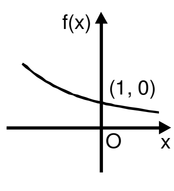
$a \rightarrow$ Base $\quad x \rightarrow$ Exponent

Case–I : $0 < a < 1$; $a = 1/2$

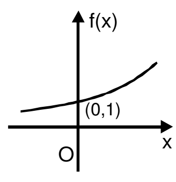
$f(x) = \left(\dfrac{1}{2}\right)^x$

Domain : $x \in R$

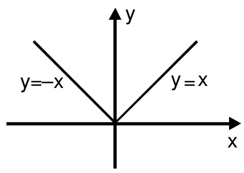
Range : $y \in (0, \infty)$



Case–II : $a > 1$



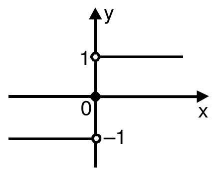
(7) Absolute value function (Modulus function) :



$$y = |x| = \begin{cases} x \ ; \ x \geq 0 \\ -x \ ; \ x < 0 \end{cases}$$

Domain : $x \in R$; Range : $y \in R^+ \cup \{0\}$

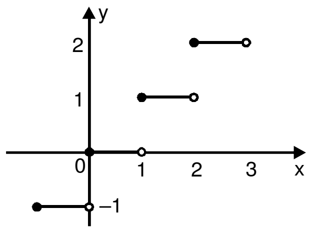
(8) Signum function :



$$y = \text{sgn}(x) = \begin{cases} 1 \ ; \ x > 0 \\ 0 \ ; \ x = 0 \\ -1 \ ; \ x < 0 \end{cases}$$

Domain : $x \in R$; Range : $y \in \{-1, 0, 1\}$

(9) Greatest integer function (step-up function) :



$$y = f(x) = [x] \begin{cases} x \ ; \ x \in I \\ \text{Greatest Integer} \ ; \ \text{otherwise} \\ \text{less than } x \end{cases}$$

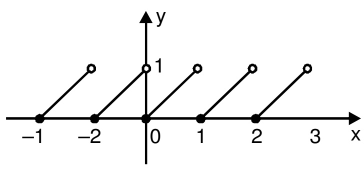
Domain : $x \in R$; Range : $y \in I$

Ex : $[2 \cdot 3] = 2$, $[5] = 5$, $[-2 \cdot 3] = -3$

Properties :

(i) $[x] \leq x < [x] + 1$ **(ii)** $[x + m] = [x] + m$; $m \in I$ **(iii)** $[x] + [-x] = \begin{cases} 0 \ ; \ x \in I \\ -1 \ ; \ x \notin I \end{cases}$

(10) Fractional part function :



$y = f(x) = \{x\} = x - [x]$

Domain : $x \in R$; Range : $[0, 1)$

Ex : $2.3 = 2 + 0.3 \rightarrow$ fractional part
$\downarrow$
Integer part

Properties :

(i) Fractional part of any integer is zero. **(ii)** $\{x + n\} = \{x\}$, $n \in I$

(iii) $\{x\} + \{-x\} = \begin{cases} 0 \ ; \ x \in I \\ 1 \ ; \ \text{otherwise} \end{cases}$

Ex.8 Find the range of the following functions : **(a)** $y = \dfrac{1}{2 + \sin 3x + \cos 3x}$ **(b)** $y = \sin^{-1}\left(\dfrac{x^2+1}{x^2+2}\right)$

Sol. (a) We have $y = \dfrac{1}{2 + \sin 3x + \cos 3x}$ i.e. $\sin 3x + \cos 3x = \dfrac{1}{y} - 2$

i.e. $\sqrt{2}\ \sin\left(3x + \dfrac{\pi}{4}\right) = \dfrac{1}{y} - 2$ i.e. $\sin\left(3x + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}\left(\dfrac{1}{y} - 2\right)$

since, $\left|\sin\left(3x+\frac{\pi}{4}\right)\right| \le 1$, therefore we have $\left|\frac{1}{y}-2\right| \le \sqrt{2}$

i.e. $-\sqrt{2} \le \frac{1}{y} - 2 \le \sqrt{2}$ i.e. $2 - \sqrt{2} \le \frac{1}{y} \le 2\sqrt{2}$

i.e. $\frac{1}{2+\sqrt{2}} \le y \le \frac{1}{2-\sqrt{2}}$ Hence, the range is $y \in \left[\frac{1}{2+\sqrt{2}}, \frac{1}{2-\sqrt{2}}\right]$.

(b) We have $\frac{x^2+1}{x^2+2} = 1 - \frac{1}{x^2+2}$

Now, we have $2 \le x^2 + 2 < \infty$ i.e. $\frac{1}{2} \ge \frac{1}{x^2+2} > 0$ i.e. $\frac{-1}{2} \le \frac{-1}{x^2+2} < 0$

i.e. $1 - \frac{1}{2} \le 1 - \frac{1}{x^2+2} < 1$ i.e. $\frac{1}{2} \le \frac{x^2+1}{x^2+2} < 1$ i.e. $\sin^{-1}\frac{1}{2} \le \sin^{-1}\left(\frac{x^2+1}{x^2+2}\right) < \sin^{-1} 1$

gives $\frac{\pi}{6} \le y < \frac{\pi}{2}$ Hence, the range is $y \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right)$.

Ex.9 Find the range of following functions : **(i)** $y = ln\,(2x - x^2)$ **(ii)** $y = \sec^{-1}(x^2 + 3x + 1)$

Sol. **(i)** using maxima–minima, we have $(2x - x^2) \in (-\infty, 1]$

For log to be defined accepted values are $2x - x^2 \in (0, 1]$ {i.e. domain (0, 1]}

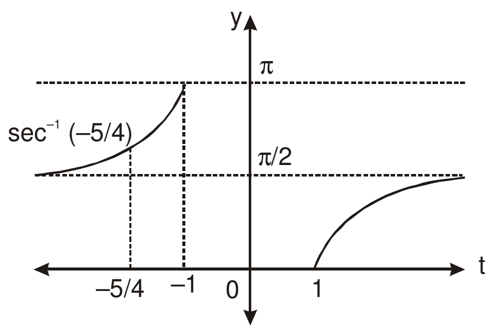
$ln\,(2x - x^2) \in (0, 1]$ $\therefore$ range is $(-\infty, 0]$

(ii) $y = \sec^{-1}(x^2 + 3x + 1)$

Let $t = x^2 + 3x + 1$ for $x \in R$ then $t \in \left[-\frac{5}{4}, \infty\right)$

but $y = \sec^{-1}(t)$ $\Rightarrow$ $t \in \left[-\frac{5}{4}, -1\right] \cup [1, \infty)$

from graph range is $y \in \left[0, \frac{\pi}{2}\right) \cup \left[\sec^{-1}\left(-\frac{5}{4}\right), \pi\right]$



Ex.10 Find the range of $y = \left[\ln(\sin^{-1}\sqrt{x^2+x+1})\right]$

Sol. We have $x^2 + x + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$ which is a positive quantity whose minimum value is 3/4.

Also, for the function $y = \left[\ln(\sin^{-1}\sqrt{x^2+x+1})\right]$ to be defined, we have $x^2 + x + 1 \le 1$

Thus, we have $\frac{3}{4} \le x^2 + x + 1 \le 1$ i.e. $\frac{\sqrt{3}}{2} \le \sqrt{x^2+x+1} \le$ i.e. $\frac{\pi}{3} \le \sin^{-1}(\sqrt{x^2+x+1}) \le \frac{\pi}{2}$

[$\because$ $\sin^{-1} x$ is an increasing function, the inequality sign remains same]

i.e. $\ln\left(\frac{\pi}{3}\right) \le \ln(\sin^{-1}\sqrt{x^2+x+1} \le \ln\left(\frac{\pi}{2}\right)$

i.e. $0.046 \le \ln(\sin^{-1}\sqrt{x^2+x+1})] = 0.$ Hence, the range is $y \in [\ell n\ \pi/3, \ell n\ \pi/2]$

Ex.11 $f : R \to R$, $f(x) = \frac{3x^2+mx+n}{x^2+1}$. If the range of this function is [– 4, 3) then find the value of $(m^2 + n^2)$.

Sol. $f(x) = \frac{3(x^2+1)+mx+n-3}{1+x^2}$; $f(x) = 3 + \frac{mx+n-3}{1+x^2}$

$y = 3 + \frac{mx+n-3}{1+x^2}$ for y to lie in [– 4, 3) $mx + n - 3 < 0 \ \forall \ x \in R$

this is possible only if m = 0 when, m = 0 then $y = 3 + \frac{n-3}{1+x^2}$

note that n – 3 < 0 (think !) n < 3 " $x \to \infty$, $y_{max} \to 3^-$

now y_{min} occurs at x = 0 (as $1 + x^2$ is minimum)

$y_{min} = 3 + n - 3 = n \Rightarrow n = -4$ so $m^2 + n^2 = 16$

Ex.12 Find the domain and range of $f(x) = \sin\left\{\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$

Sol. $\sqrt{4-x^2}$ is positive and $x^2 < 4 \Rightarrow -2 < x < 2$

1 – x should also be positive. $\therefore$ x < 1

Thus the domain of $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$ is –2 < x < 1 sine being defined for all values, the domain of sin $\left\{\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right\}$ is the same as the domain of $\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)$

To study the range. Consider the function $\frac{\sqrt{4-x^2}}{1-x}$

As x varies from –2 to 1, $\frac{\sqrt{4-x^2}}{1-x}$ varies in the open interval $(0, \infty)$ and hence $\ell n\frac{\sqrt{4-x^2}}{1-x}$ varies from $-\infty$ to $+\infty$. Therefore the range of $\sin\left(\ell n\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right)$ is (–1, +1)

Ex.13 Find the range of the function $f(x) = \sin^{-1} \dfrac{\sqrt{1+x^4}}{1+5x^{10}}$.

Sol. Consider $g(x) = \dfrac{\sqrt{1+x^4}}{1+5x^{10}}$. Also g(x) is positive $\forall\ x \in R$ and g(x) is continuous $\forall\ x \in R$ and

$g(0) = 1$ and $\lim\limits_{x\to\infty} g(x) = 0$

$\Rightarrow$ g(x) can take all values from (0, 1] $\Rightarrow$ Range of $f(x) = \sin^{-1}(g(x))$ is $\left(0, \dfrac{\pi}{2}\right]$.

Ex.14 $f(x) = \cos^{-1}\{\log[\sqrt{[x^3+1]}]\}$, find the domain and range of f(x) (where [$*$] denotes the greatest integer function).

Sol. If $\cos^{-1} x = \theta$, then $-1 \le x \le 1$ $\therefore$ $-1 \le \log[\sqrt{[x^3+1]}] \le 1$ $\Rightarrow$ $e^{-1} \le [\sqrt{[x^3+1]}] \le e$

$0.37 \le [\sqrt{[x^3+1]}] \le 2.7$ $\therefore$ $1 \le \sqrt{[x^3+1]} < 3 \Rightarrow 1 \le [x^3 + 1] < 9$

$1 \le [x^3] + 1 < 9$ $\quad 0 \le [x^3] < 8$ $\quad\therefore\quad 0 \le x < 2$

$\therefore$ Domain of $f(x) = D_r$ in $x \in [0, 2)$ Range of f(x) When $0 \le x < 2$

Then $1 \le x^3 + 1 < 9$ $\therefore$ $1 \le [x^3 + 1] \le 8$ $\Rightarrow$ $1 \le \sqrt{[x^3+1]} \le 2\sqrt{2} \Rightarrow 1 \le \sqrt{[x^3+1]} \le 2.8$

Case I : $1 \le \sqrt{[x^3+1]} < 2$ then $[\sqrt{[x^3+1]}] = 1$ **Case II :** $2 \le \sqrt{[x^3+1]} \le 2.8$ then $[\sqrt{[x^3+1]}] = 2$

$\therefore$ Range in $\cos^{-1}\{\log 1\}$ and $\cos^{-1}\{\log 2\}$ $\therefore$ R_f is $(\pi/2, \cos^{-1}(\log 2))$

Ex.15 Find the range of the following functions

(i) $f(x) = \log_e(\sin x^{\sin x} + 1)$ where $0 < x < \pi/2$.

(ii) $f(x) = \log_e(2\sin x + \tan x - 3x + 1)$ where $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{3}$

Sol. **(i)** $0 < x < \pi/2$ $\Rightarrow$ $0 < \sin x < 1$

$\therefore$ Range of $\log_e(\sin x^{\sin x} + 1)$ for $0 < x < \pi/2$ = Range of $\log_e(x^x + 1)$ for $0 < x < 1$

Let $h(x) = x^x + 1 = e^{x\log_e x} + 1$

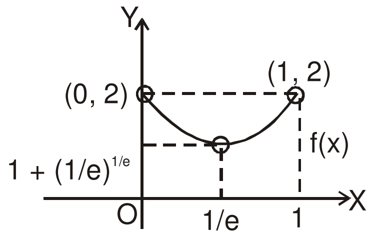
$\therefore$ $h'(x) = e^{x\log_e x}(1 + \log_e x)$ $\Rightarrow$ $h'(x) > 0$ for $x > 1/e$ and $h'(x) < 0$ for $x < 1/e$

$\therefore$ h(x) has a minima at x = 1/e

Also $\lim_{x\to0^+} h(x) = 1 + e^{\lim_{x\to0}\left(\frac{\ln x}{1/x}\right)} = 1 + e^{\lim_{x\to0}\left(\frac{1/x}{-1/x^2}\right)} = 1 + e^0 = 2$ and $\lim_{x\to1^-} h(x) = 2$

$\therefore \quad 0 < x < 1 \quad \Rightarrow \quad 1 + \left(\frac{1}{e}\right)^{\frac{1}{e}} < (x^x + 1) < 2$

$\Rightarrow \quad \log_e\left(1+\left(\frac{1}{e}\right)^{\frac{1}{e}}\right) < \log_e (x^x + 1) < \log_e 2$



$\therefore \quad$ Range of f(x) = $\left(\log_e\left(1+e^{\frac{-1}{e}}\right), \log_e 2\right)$

(ii) Let h(x) = (2 sin x + tan x – 3x +1) $\Rightarrow$ h′(x) = (2 cos x + sec² x – 3)

$= \dfrac{2\cos^3 x - 3\cos^2 x + 1}{\cos^2 x} \qquad \therefore \quad h'(x) > 0 \Rightarrow 2\cos^3 x - 3\cos^2 x + 1 > 0$

$(\cos x - 1)^2\left(\cos x + \frac{1}{2}\right) > 0 \ \forall\ x \in [\pi/6, \pi/3] \qquad \Rightarrow$ h(x) is an increasing function of x

$\Rightarrow \quad h(\pi/6) \le h(x) \le h(\pi/3) \quad \Rightarrow \quad \log_e\left(2+\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right) \le \log_e h(x) \le \log_e (1 + 2\sqrt{3} - \pi)$

$\therefore \quad$ Range of f(x) is $\left[\log_e\left(2+\frac{1}{\sqrt{3}}-\frac{\pi}{2}\right), \log(1+2\sqrt{3}-\pi)\right]$

(11) Equal or Identical Functions : Two functions f & g are said to be equal if :

(i) The domain of f = The domain of g $\Rightarrow D_f = D_g$

(ii) The range of f = The range of g $\Rightarrow R_f = R_g$

(iii) f(x) = g(x), $\forall\, x \in$ their common domain.

Ex.16 Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ then D_f : R – {0} and D_g : R – {0}

$\therefore D_f = D_g \Rightarrow$ Hence both functions are identical

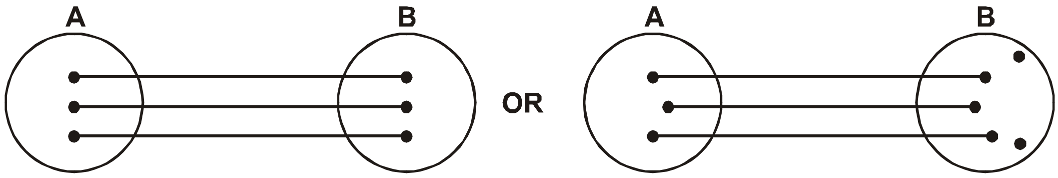
Ex.17 Let f(x) = sin x and $g(x) = \frac{1}{\operatorname{cosec} x}$ then $D_f : x \in R$ and $D_g : x \in R - \{n\pi\}$

$\therefore D_f \ne D_g \Rightarrow$ Hence both functions are non-identical

D. CLASSIFICATION OF FUNCTIONS

(1) One – One Function (Injective mapping) : A function $f : A \to B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagrammatically an injective mapping can be shown as



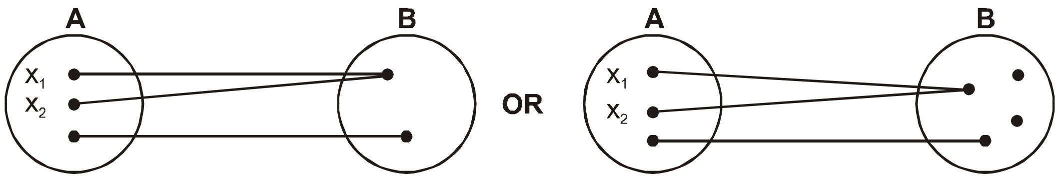
Remark :

(i) Any function which is entirely increasing or decreasing in its domain, is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one.

(2) Many–One function : A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B . Thus $f : A \to B$ is many one if for $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagrammatically a many one mapping can be shown as



Remark :

(i) A continuous function f(x) which has atleast one local maximum or local minimum, is many–one. In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one.

(ii) If a function is one–one, it cannot be many–one and vice versa.

(iii) If f and g both are one-one, then fog and gof would also be one-one (if they exist).

Ex.18 Show that the function $f(x) = \dfrac{x^2 - 8x + 18}{x^2 + 4x + 30}$ is not one-one.

Sol. Test for one-one function

A function is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Now $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1^2 - 8x_1 + 18}{x_1^2 + 4x_1 + 30} = \dfrac{x_2^2 - 8x_2 + 18}{x_2^2 + 4x_2 + 30}$

$\Rightarrow \quad 12x_1^2 x_2 - 12x_1 x_2^2 + 12x_1^2 - 12x_2^2 - 312x_1 + 312x_2 = 0$

$\Rightarrow \quad (x_1 - x_2)\{12x_1x_2 + 12(x_1 + x_2) - 312\} = 0 \quad \Rightarrow \quad x_1 = x_2$ or $x_1 = \dfrac{26 - x_2}{1 + x_2}$

Since $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$ alone, f(x) is not a one-one function.

Ex.19 Let f be an injective function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)\ \forall\ x, y \in R$.
If $f(4) = 65$ and $f(0) \neq 2$, then show that $f(x) - 1 = x^3\ \forall\ x \in R$.

Sol. Given that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ ....(i)
Putting x = y = 0 in equation (i), we get $f(0)\,f(0) + 2 = f(0) + f(0) + f(0)$
or $(f(0))^2 + 2 = 3f(0)$ or $(f(0) - 2)\,(f(0) - 1) = 0$ or $f(0) = 1$ ($\because f(0) \neq 2$) ....(ii)
Again putting x = y = 1 in equation (i) and repeating the above steps, we get
$(f(1) - 2)\,(f(1) - 1) = 0$
But $f(1) \neq 1$ as f(x) is injective. $\therefore$ $f(1) = 2$ ....(iii)
Now putting y = 1/x in equation (i), we get

$$f(x)\,f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + f(1) \quad \text{or} \quad f(x)\,f\left(\frac{1}{x}\right) + 2 = f(x) + f\left(\frac{1}{x}\right) + 2$$

$$\text{or} \quad f(x)\,f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \quad \text{or} \quad f(x)\,f\left(\frac{1}{x}\right) - f(x) - f\left(\frac{1}{x}\right) - 1 + 1 = 0$$

$$\text{or} \quad f(x)\left\{f\left(\frac{1}{x}\right) - 1\right\} - 1.\left\{f\left(\frac{1}{x}\right) - 1\right\} = 1 \quad \text{or} \quad \{f(x) - 1\}\left\{f\left(\frac{1}{x}\right) - 1\right\} = 1 \quad ...(iv)$$

$$\text{Let} \quad f(x) - 1 = g(x) \quad \Rightarrow \quad f\left(\frac{1}{x}\right) - 1 = g\left(\frac{1}{x}\right)$$

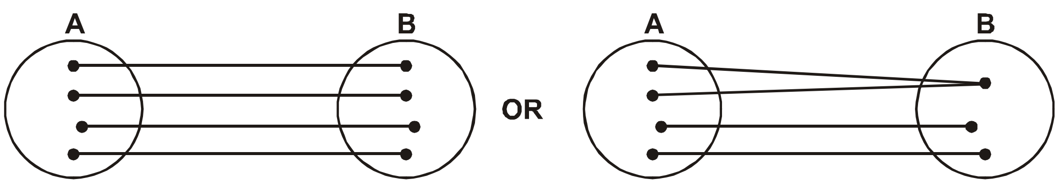
$\therefore$ from equation (iv), we get $g(x)\,g\left(\frac{1}{x}\right) = 1$ which is only possible when

$g(x) = \pm x^n$ $\therefore$ $f(x) = \pm x^n + 1$ or $f(x) = \pm x^n + 1$ or $65 = \pm 4^n + 1$

or $4^n = 64 = (4)^3$ $\therefore$ $n = 3$ $\therefore$ $f(x) = x^3 + 1$ or $f(x) - 1 = x^3$ (neglecting negative sign)

(3) Onto-function (Surjective mapping) : If the function $f : A \to B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B . Thus $f : A \to B$ is surjective iff $\forall\ b \in B,\ \exists$ some $a \in A$ such that $f(a) = b$.

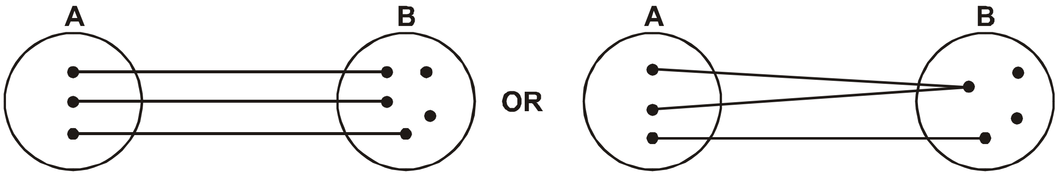
Diagramatically surjective mapping can be shown as



Note that : if range $\equiv$ co–domain, then f(x) is onto.

(4) Into function : If $f : A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into.

Diagramatically into function can be shown as



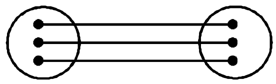
Remark :

(i) If a function is onto, it cannot be into and vice versa .

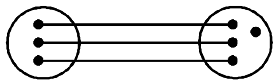
(ii) If f and g are both onto, then gof or fog may or may not be onto.

Thus a function can be one of these four types :

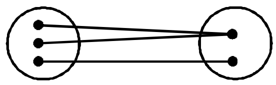
(a) one–one onto (injective & surjective)



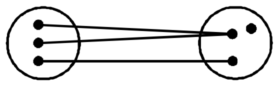
(b) one–one into (injective but not surjective)



(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Remark :

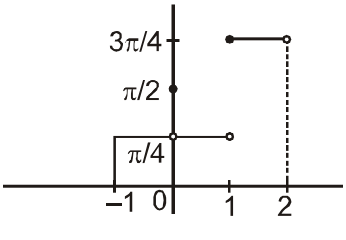
(i) If f is both injective & surjective, then it is called a **Bijective** function. Bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one-one.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Ex.20 A function is defined as , $f : D \to R$ $f(x) = \cot^{-1}(\text{sgn } x) + \sin^{-1}(x - \{x\})$ (where $\{x\}$ denotes the fractional part function) Find the largest domain and range of the function. State with reasons whether the function is injective or not . Also draw the graph of the function.

Sol. D $[-1, 2)$, $R = \left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ f is many one



Ex.21 Find the linear function(s) which map the interval $[0, 2]$ onto $[1, 4]$.

Sol. Let $f(x) = ax + b$

$f(0) = 1$ & $f(2) = 4 \Rightarrow b = 1$ & $a = \frac{3}{2}$

or $f(0) = 4$ & $f(2) = 1 \Rightarrow b = 4$ & $a = -\frac{3}{2}$

Ans. : $f(x) = \frac{3x}{2} + 1$ or $f(x) = 4 - \frac{3x}{2}$

Ex.22 (i) Find whether $f(x) = x + \cos x$ is one–one.

(ii) Identify whether the function $f(x) = -x^3 + 3x^2 - 2x + 4$; $R \to R$ is ONTO or INTO

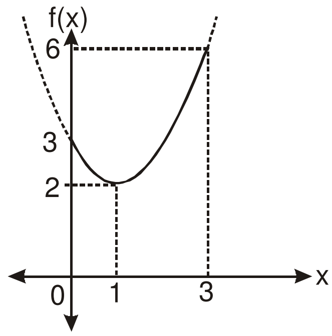
(iii) $f(x) = x^2 - 2x + 3$; $[0, 3] \to A$. Find whether f(x) is injective or not. Also find the set A, if f(x) is surjective.

Sol. (i) The domain of f(x) is R. $f'(x) = 1 - \sin x$.

$\therefore$ $f'(x) \geq 0 \ \forall \ x \in$ complete domain

and equality holds at discrete points only

$\therefore$ f(x) is strictly increasing on R. Hence f(x) is one-one.



(ii) As codomain $\equiv$ range, therefore given function is ONTO

(iii) $f'(x) = 2(x - 1)$; $0 \leq x \leq 3$

$$\therefore f'(x) = \begin{cases} -\text{ve} & ; \ 0 \leq x < 1 \\ +\text{ve} & ; \ 1 < x < 3 \end{cases}$$

$\therefore$ f(x) is a non monotonic continuous function. Hence it is not injective.

For f(x) to be surjective, A should be equal to its range. From graph,, range is [2, 6]

$\therefore$ $A \equiv [2, 6]$

Ex.23 If f and g be two linear functions from $[-1, 1]$ onto $[0, 2]$ and $\phi : R^+ - \{-1, 1\} \to R$ be defined by

$\phi(x) = \frac{f(x)}{g(x)}$, then show that $\left|\phi(\phi(x)) + \phi\left(\phi\left(\frac{1}{2}\right)\right)\right| \geq 2$.

Sol. Let h be a linear function from $[-1, 1]$ onto $[0, 2]$.

Let $h(x) = ax + b$, then $h'(x) = a$

If $a > 0$, then h(x) is an increasing function & $h(-1) = 0$ and $h(1) = 2 \Rightarrow -a + b = 0$ and $a + b = 2$

$\Rightarrow a = 1$ & $b = 1$. Hence $h(x) = x + 1$.

If $a < 0$, then h(x) is a decreasing function & $h(-1) = 2$ and $h(1) = 0 \Rightarrow -a + b = 2$ and $a + b = 0$

$\Rightarrow a = -1$ & $b = 1$. Hence $h(x) = 1 - x$

Now according to the question $f(x) = 1 + x$ & $g(x) = 1 - x$

or $f(x) = 1 - x$ & $g(x) = 1 + x$ $\therefore$ $\phi(x) = \frac{f(x)}{g(x)} = \frac{1-x}{1+x}$ or $\frac{1+x}{1-x}$

Case-I : When $\phi(x) = \left(\frac{1-x}{1+x}\right)$, $x \neq -1$; $\phi\left(\phi\left(\frac{1}{x}\right)\right) = \frac{1}{x}$

Case-II : When $\phi(x) = \left(\frac{1+x}{1-x}\right)$, $x \neq 1$ $\phi\left(\phi\left(\frac{1}{x}\right)\right) = -x.$

In both cases, $|\phi(f(x)) + \phi(\phi(1/x))| = \left|x + \frac{1}{x}\right|$ (where $x > 0$) $= \left|\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2\right| \geq 2$

E. FUNCTIONAL EQUATION

Functional Equation is an equation where the unknown is a function. On solving such an equation we obtain one or more functions as solutions. If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\, \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) . f(y) \Rightarrow f(x) = x^n$, $n \in R$

(iii) $f(x + y) = f(x) . f(y) \Rightarrow f(x) = a^{kx}$, $a > 0$

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Ex.24 If $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$ and $f(0) = 1$ $\forall$ x, y $\in$ R. Determine f(x).

Sol. Given $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2$

Putting x = y = 0; then $f(1) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1 + 1)^2 = 2^2$

Again putting x = 0, y = 1 Then $f(2) = (\sqrt{f(0)} + \sqrt{f(0)})^2 = (1 + 2)^2 = 3^2$

and for x = 1, y = 1; $f(3) = (\sqrt{f(1)} + \sqrt{f(1)})^2 = (2 + 2)^2 = 4^2$ Similarly, $f(x) = (x + 1)^2$

Ex.25 Let $f : R - \{2\} \to R$ function satisfying the following functional equation,

$2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80$, $\forall\, x \in R - \{2\}$. Determine f(x).

Sol. We have, $f(x) = -\frac{3}{2} f\left(\frac{2x+29}{x-2}\right) + 50x + 40$ ...(i)

Replacing x by $\frac{2x+29}{x-2}$ in the given functional equation we get,

$$f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2} f\left(\frac{2\left(\frac{2x+29}{x-2}\right)+29}{\left(\frac{2x+29}{x-2}\right)-2}\right) + 50\left(\frac{2x+29}{x-2}\right) + 40$$

$$\Rightarrow \quad f\left(\frac{2x+29}{x-2}\right) = -\frac{3}{2}f(x) + 50\left(\frac{2x+29}{x-2}\right) + 40 \qquad ...(ii)$$

putting (ii) in (i), we get,

$$f(x) = \frac{9}{4}f(x) - 75\left(\frac{2x+29}{x-2}\right) - 60 + 50x + 40 \quad \Rightarrow \quad \frac{9}{4}f(x) - f(x) = 20 - 50x + 75\left(\frac{2x+29}{x-2}\right)$$

$$\Rightarrow \quad \frac{5}{4}f(x) = 20 - 50x + 75\left(\frac{2x+29}{x-2}\right) \quad \Rightarrow \quad f(x) = 16 - 40x + 60\frac{(2x+29)}{(x-2)}$$

Ex.26 Let f be a function from the set of positive integers to the set of real numbers i.e., $f : N \to R$ such that **(i)** $f(1) = 1$; **(ii)** $f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n+1)f(n)$ for $n \geq 2$ then find the value of $f(1994)$.

Sol. Given $f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n+1)f(n)$ ...(1)

Replacing n by (n + 1) then

$f(1) + 2f(2) + 3f(3) + ... + nf(n) = n(n+1)f(n+1) = (n+1)(n+2)f(n+1)$ ...(2)

Subtracting (1) from (2) then we get

$(n+1)f(n+1) = (n+1)(n+2)f(n+1) - n(n+1)f(n) \quad \Rightarrow \quad nf(n) = (n+1)f(n+1)$

From which we conclude that $2f(2) = 3f(3) = 4f(4) = ... = nf(n)$

Substituting the value of 2f(2), 3f(3), ... in terms of nf(n) in (1), we have

$f(1) + (n-1)nf(n) = n(n+1)f(n) \quad \Rightarrow f(1) = 2n f(n) \quad \therefore \ f(n) = \frac{f(1)}{2n} = \frac{1}{2n} \quad (\because f(1) = 1)$

$$\Rightarrow \quad f(1994) = \frac{1}{2.1994} = \frac{1}{3988}$$

Ex.27 Let f be a function satisfying $2f(xy) = \{f(x)\}^y + \{f(y)\}^x$ and $f(1) = k \neq 1$. Prove that $(k-1)\sum_{r=1}^{n} f(r) = k^{n+1} - k$

Sol. Given that $2f(xy) = f(x))^y + (f(y))^x$ ...(1)

replacing y by 1, we get $2f(x) = f(x) + (f(1))^x$ or $f(x) = f(1)^x = k^x \{\because f(1) = k\}$ ....(2)

$$\therefore \sum_{r=1}^{n} f(r) = f(1) + f(2) + ... + f(2) = k^1 + k^2 + k^3 + ... + k^n$$

$$= \frac{k(k^n - 1)}{(k-1)} = \frac{k^{n+1} - k}{(k-1)} \quad \text{or} \quad (k-1)\sum_{r=1}^{n} f(r) = (k^{n+1} - k)$$

Ex.28 Determine all the functions $f : R \to R$ such that $f(x)f(y) - f(xy) = x + y$ for all $x, y \in R$ ...(1)

Sol. Put $x = y = 0$ in (1). Hence $f(0)f(0) - f(0) = 0$. This implies that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$ then $f(x)f(0) - f(0) = x + 0$. Hence, $x = 0$ for all $x \in R$, a contradiction. Hence, $f(0) = 1$.

Substituting $y = 0$, in (1) we get $f(x) . f(0) - f(0) = x + 0 \Rightarrow f(x) = x + 1$ is the only solution of (1).

F. COMPOSITE FUNCTIONS

Let f: $X \to Y_1$ and g: $Y_2 \to Z$ be two functions and the set D = {x ∈ X: f(x) ∈ Y_2}. If D ≢ φ, then the function h defined on D by h(x) = g{f(x)} is called composite function of g and f and is denoted by gof. It is also called function of a function.

Remark : Domain of gof is D which is a subset of X (the domain of f). Range of gof is a subset of the range of g. If D = X, then $f(X) \subseteq Y_2$.

Properties of composite functions :

(i) The composite of functions is not commutative i.e. gof ≠ fog.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

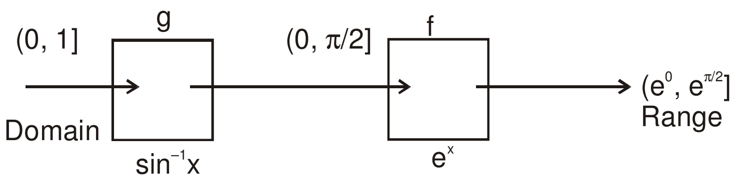
Ex.29 Let f(x) = e^x ; $R^+ \to R$ and g(x) = $\sin^{-1} x$; $[-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find domain and range of fog (x)

Sol. Domain of f(x) : (0, ∞), Range of g(x) : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The values in range of g(x) which are accepted by f(x) are $\left(0, \frac{\pi}{2}\right]$

$\Rightarrow \quad 0 < g(x) \le \frac{\pi}{2} \qquad 0 < \sin^{-1} x \le \frac{\pi}{2} \qquad 0 < x \le 1$

Hence domain of fog(x) is x ∈ (0, 1]



Therefore Domain : (0, 1], Range : $(1/ e^{\pi/2}]$

Ex.30 Let f(x) = $x^2 - 1$ and g(x) = $\begin{cases} [|f(|x|)|], & x \in (-1, 0) \cup (0, 1) \\ 1, & \text{otherwise} \end{cases}$. Then find the range of $\ell n([|g(x)|])$

(where [∗] denotes the greatest integer function)

Sol. | f (|x|) | = $1 - x^2$, x ∈ (–1, 0) ∪ (0, 1) ⇒ [| f (|x|) |] = 0, x ∈ (–1, 0) ∪ (0, 1)

Also [f(x)] = –1, x ∈ (–1, 0) ∪ (0, 1) ⇒ | [f(x)] | = 1, x ∈ (–1, 0) ∪ (0, 1)

∴ g(x) = 1, x ∈ R ⇒ range of ℓn([| g(x) |]) = {0}.

Ex.31 Let f(x) = $\frac{x-1}{x+1}$, $f^2(x)$ = f{f(x)}, f^3 (x) = f{f^2(x)},.....f^{k+1} (x) = f{f^k(x)}. for k = 1, 2, 3,...., Find f^{1998} (x).

Sol. f(x) = $\frac{x-1}{x+1}$, $f^2(x)$ = f{f(x)} = $\frac{f-1}{f+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{-1}{x}$, $f^3(x)$ = f{f(x)} = $\frac{f^2(x)-1}{f^2(x)+1} = \frac{\frac{-1}{x}-1}{\frac{-1}{x}+1} = \frac{x+1}{x-1}$,

$f^4 = f\{f^3(x)\} = \dfrac{f^3(x)-1}{f^3(x)+1} = \dfrac{\frac{x+1}{x-1}-1}{\frac{x+1}{x-1}+1} = x,$ $\qquad f^5(x) = f\{f^4(x)\} = \dfrac{x+1}{x-1} = f(x)$

Thus, we can see that $f^k(x)$ repeats itself at intervals of k = 4.

Hence, we have $f^{1998}(x) = f^2(x) = \dfrac{-1}{x}$ $\qquad [\therefore 1998 = 499 \times 4 + 2]$

Ex.32 Let $g : R \to R$ be given by g(x) = 3 + 4x. If $g^n(x)$ = gogo....og(x), show that $f^n(x) = (4^n - 1) + 4^n x$ if $g^{-n}(x)$ denotes the inverse of $g^n(x)$.

Sol. Since g(x) = 3 + 4x

$\therefore g^2(x) = (gog)(x) = g\{g(x)\} = g(3+4x) = 3 + 4(3+4x)$ or $g^2(x) = 15 + 4^2x = (4^2-1) + 4^2x$

Now $g^3(x) = (gogog)\,x = g\{g^2(x)\} = g(15 + 4^2x) = 3 + 4(15 + 4^2x) = 63 + 4^3x = (4^3-1) + 4^3x$

Similarly we get $g^n(x) = (4^n - 1) + 4^n x$

Now leg $g^n(x) = y \Rightarrow x = g^{-n}(y)$ ...(1)

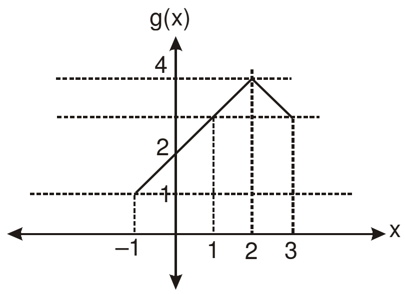
$\therefore y = (4^n - 1) + 4^n x$ or $x = (y + 1 - 4^n)4^{-n}$ ...(2)

From (1) and (2) we get $g^{-n}(y) = (y + 1 - 4^n)\,4^{-n}$. Hence $g^{-n}(x) = (x + 1 - 4^n)\,4^{-n}$

Ex.33 If f(x) = | |x − 3| − 2 | ; $0 \le x \le 4$ and g(x) = 4 − |2 − x| ; $-1 \le x \le 3$ then find fog(x).

Sol. $g(x) = \begin{cases} 2+x & -1 \le x < 2 \\ 6-x & 2 \le x \le 3 \end{cases}$ $\qquad \therefore fog(x) = \begin{cases} 1-g(x) & 0 \le g(x) < 1 \\ g(x)-1 & 1 \le g(x) < 3 \\ 5-g(x) & 3 \le g(x) \le 4 \end{cases} = \begin{cases} 1-g(x) & \text{for no value} \\ g(x)-1 & -1 \le x < 1 \\ 5-g(x) & 1 \le x \le 3 \end{cases}$

$= \begin{cases} 2+x-1 & -1 \le x < 1 \\ 5-(2+x) & 1 \le x < 2 \\ 5-(6-x) & 2 \le x \le 3 \end{cases} = \begin{cases} x+1 & -1 \le x < 1 \\ 3-x & 1 \le x < 2 \\ x-1 & 2 \le x \le 3 \end{cases}$



Ex.34 Prove that f(n) = 1 − n is the only integer valued function defined on integers such that

(i) f(f(n)) = n for all $n \in Z$ and **(ii)** f(f(n + 2) + 2) = n for all $n \in Z$ and **(iii)** f(0) = 1.

Sol. The function f(n) = 1 − n clearly satisfies conditions (i), (ii) and (iii). Conversely, suppose a function $f : Z \to Z$ satisfies (i), (ii) and (iii). Applying f to (ii) we get, f(f(f(n + 2) + 2))) = f(n)

and this gives because of (i), f(n + 2) + 2 = f(n), ...(1)

for all $n \in Z$. Now using (1) it is easy to prove by induction on n that for all $n \in Z$,

$f(n) = \begin{cases} f(0) - n \text{ if n is even} \\ f(1) + 1 - n \text{ if n is odd} \end{cases}$

Also by (iii), f(0) = 1. Hence by (i), f(1) = 0. Hence f(n) = 1 − n for all $n \in Z$.

G. GENERAL DEFINITION

(1) Identity function : A function $f : A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A & denoted by I_A. **Ex :** $f : R^+ \to R^+$; $f(x) = e^{\ell n\, x}$ and $f : R \to R$; $f(x) = \ell n\, e^x$
Every Identity function is a bijection.

(2) Constant function : A function $f : A \to B$ is said to be constant function. If every element of set A has the same functional image in set B i.e. $f : A \to B$; $f(x) = c \ \forall \ x \in A$ & $c \in B$ is called constant function.

(3) Homogeneous function : A function is said to be homogeneous w.r.t. any set of variables when each of its term is of the same degree w.r.t. those variables.

(4) Bounded Function : A function $y = f(x)$ is said to be bounded if it can be express is the form of $a \le f(x) \le b$ where a and b are finite quantities.
Ex : $-1 \le \sin x \le 1$; $0 \le \{x\} < 1$; $-1 \le \text{sgn}(x) \le 1$ but e^x is not bounded.
Ex : Any function having singleton range like constant function.

(5) Implicit function & Explicit function : If y has been expressed entirely in terms of 'x' then it is called an explicit function.
If x & y are written together in the form of an equation then it is known as implicit equation corresponding to each implicit equation there can be one, two or more explicit function satisfying it
Ex : $y = x^3 + 4x^2 + 5x$ → Explicit function
Ex : $x + y = 1$ → Implicit equation
Ex : $y = 1 - x$ → Explicit function

H. EVEN & ODD FUNCTIONS

Function must be defined in symmetric interval $[-x, x]$
If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.
If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Remark :

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .
(b) A function may be neither even nor odd.
(c) Inverse of an even function is not defined.
(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin .
(e) A function (whose domain is symmetric about origin) can be expressed as a sum of an even & an odd function. e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$
(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.
(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd and other even then f.g will be odd.

Ex.35 Which of the following functions is odd ?

(A) $\text{sgn } x + x^{2000}$ (B) $|x| - \tan x$ (C) $x^3 \cot x$ (D) $\text{cosec } x^{55}$

Sol. Let's name the function of the parts (A), (B), (C) and (D) as f(x), g(x), h(x) & $\phi(x)$ respectively. Now

(A) $f(-x) = \text{sgn}(-x) + (-x)^{2000} = -\text{sgn } x + x^{2000} \neq f(x) \;\&\; \neq -f(x)$ $\therefore$ f is neither even nor odd.

(B) $g(-x) = |-x| - \tan(-x) = |x| + \tan x$ $\therefore$ g is neither even nor odd.

(C) $h(-x) = (-x)^3 \cot(-x) = -x^3(-\cot x) = x^3 \cot x = h(x)$ $\therefore$ h is an even function

(D) $\phi(-x) = \text{cosec}(-x)^{55} = \text{cosec}(-x^{55}) = -\text{cosec } x^{55} = -\phi(x)$ $\therefore$ ϕ is an odd function.

Alternatively

(A) $f(x) = \text{sgn}(x) + x^{2000} = O + E$ = neither E nor O

(B) $g(x) = E - O$ = Neither E nor O

(C) $h(x) = O \times O = E$

(D) $f(-x) = O \circ O = O$ $\therefore$ **(D)** is the correct option

Ex.36 $f(x) = (\tan x^5)\, e^{x^3 \text{sgn}\, x^7}$ is

(A) an even function (B) an odd function

(C) neither even nor odd function (D) none of these

Sol. $f(x) = (\tan\ (x^5))\ e^{x^3\ \text{sgn}\ (x^7)}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$O \quad (O) \quad e^{O \times O\ (O)}$ $= O \times e^{O \times O} = O \times e^{E}$ $= O \times E = O$

Ex.37 Let f: $[-2, 2] \to R$ be a function if $f(x) = \begin{cases} x\tan x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} < x \le 2 \end{cases}$ Define f for $x \in [-2, 0]$ so that

(i) f is an odd function **(ii)** f is an even function (where [$*$] denotes the greatest integer function)

Sol. Since $f(x) = \begin{cases} x\tan x, & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[x], & \frac{\pi}{2} < x \le 2 \end{cases}$ $\therefore f(-x) = \begin{cases} (-x)\tan(-x), & 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2}[-x], & \frac{\pi}{2} < -x \le 2 \end{cases}$ or $f(-x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$

(i) If f is an odd function then $f(x) = -f(-x) = \begin{cases} -x\tan x, & -\frac{\pi}{2} \le x < 0 \\ -\frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$

(ii) If f is an even function $\therefore$ $f(x) = f(-x) = \begin{cases} x\tan x, & -\frac{\pi}{2} \le x < 0 \\ \frac{\pi}{2}[-x], & -2 \le x < -\frac{\pi}{2} \end{cases}$

Ex.38 Let $f(x) = e^x + \sin x$ be defined on the interval [–4, 0]. Find the odd and even extension of f(x) in the interval [–4, 4].

Sol. **Odd Extension :** Let g_0 be the odd extension of f(x), then

$$g_0(x) = \begin{cases} f(x) & ; \ x \in [-4,0] \\ -f(-x) & ; \ x \in [0,4] \end{cases} = \begin{cases} e^x + \sin x & ; \ x \in [-4,0] \\ -e^{-x} + \sin x & ; \ x \in [0,4] \end{cases}$$

Even Extension : Let g_e be the even extension of f(x), then

$$g_e(x) = \begin{cases} f(x) & ; \ x \in [-4,0] \\ f(-x) & ; \ x \in [0,4] \end{cases} = \begin{cases} e^x + \sin x & ; \ x \in [-4,0] \\ e^{-x} - \sin x & ; \ x \in [0,4] \end{cases}$$

I. PERIODIC FUNCTION

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x + T) = f(x)$, for all values of x and x + T within the domain of f(x). The least positive period is called the principal or fundamental period of f.

e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π.

Remark :

(a) A constant function is always periodic, with no fundamental period.

(b) If f(x) has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(c) if f(x) has a period T then f(ax + b) has a period T/a (a > 0).

(d) If f(x) has a period T_1 & g(x) also has a period T_2 then period of f(x) ± g(x) or $\frac{f(x)}{g(x)}$ is L.C.M of T_1 & T_2 provided their L.C.M. exists. However that L.C.M. (if exists) need not to be fundamental period. If L.C.M. does not exists then f(x) ± g(x) or f(x) . g(x) or $\frac{f(x)}{g(x)}$ is nonperiodic e.g. |sin x| has the period π, |cos x| also has the period π

$\therefore$ |sin x| + |cos x| also has a period π. But the fundamental period of |sin x| + |cos x| is $\pi/2$.

(e) If g is a function such that gof is defined on the domain of f and f is periodic with T, then gof is also periodic with T as one of its periods. Further if

\# g is one-one, then T is the period of gof

\# g is also periodic with T′ as the period and the range of f is a subset of [0, T′], then T is the period of gof

(f) Inverse of a periodic function does not exist.

Ex.39 Find period of the following functions

(i) $f(x) = \sin \frac{x}{2} + \cos \frac{x}{3}$ **(ii)** $f(x) = \{x\} + \sin x$

(iii) $f(x) = \cos x \cdot \cos 3x$ **(iv)** $f(x) = \sin \frac{3x}{2} - \cos \frac{x}{3} - \tan \frac{2x}{3}$.

Sol. **(i)** Period of sin x/2 is 4π while period of cos x/3 is 6π. Hence period of sin x/2 + cos x/3 is 12π {L.C.M. or 4 & 6 is 12}

(ii) Period of sin x = 2π ; Period of {x} = 1; but L.C.M. of 2π & 1 is not possible $\therefore$ it is aperiodic

(iii) $f(x) = \cos x \cdot \cos 3x$; Period of f(x) is L.C.M. of $\left(2\pi, \frac{2\pi}{3}\right) = 2\pi$

but 2π may or may not be the fundamental period. The fundamental period can be $\frac{2\pi}{n}$, where $n \in N$. Hence cross-checking for n = 1, 2, 3..... we find π to be fundamental period
$f(\pi + x) = (-\cos x)(-\cos 3x) = f(x)$

(iv) Period of f(x) is L.C.M. of $\frac{2\pi}{3/2}, \frac{2\pi}{1/3}, \frac{\pi}{3/2}$ = L.C.M. of $\frac{4\pi}{3}, 6\pi, \frac{2\pi}{3} = 12\pi$

Ex.40 Prove that sin (1/x), ($x \neq 0$) is a non-periodic function

Sol. Let f(x) = sin (1/x) be periodic with period T, $T \neq 0, T > 0$ $\therefore$ $f(x + T) = f(x)$

$\Rightarrow \quad \sin\left(\frac{1}{x+T}\right) = \sin\left(\frac{1}{x}\right) \quad \Rightarrow \quad \frac{1}{x+T} = n\pi + (-1)^n \frac{1}{x}$ ...(1)

Put x = T and x = 2T in (1), then $\frac{1}{2T} = n\pi + (-1)^n \frac{1}{T}$ ...(2)

and $\frac{1}{3T} = np + (-1)^n \frac{1}{2T}$ ...(3)

Subtracting (3) from (2), we get $\frac{1}{6T} = (-1)^n \cdot \frac{1}{2T}$ or $\frac{1}{3} = (-1)^n$ which is impossible.

Hence sin (1/x), ($x \neq 0$) is non-periodic function.

Ex.41 If f(x) = sin x + cos ax is a periodic function, show that a is a rational number.

Sol. Given f(x) = sin x + cos ax

$\because$ Period of sin x = $\frac{2\pi}{1}$ and period of cos ax = $\frac{2\pi}{a}$

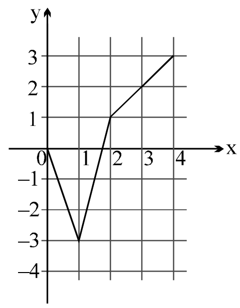
Hence period of f(x) = L.C.M. or $\left\{\frac{2\pi}{1}, \frac{2\pi}{a}\right\} = \frac{\text{L.C.M. of } \{2\pi, 2\pi\}}{\text{H.C.F. of } \{1, a\}} = \frac{2\pi}{k}$

where k = H.C.F. of 1 and a

$\therefore$ $\frac{1}{k}$ = integer = q (say), ($\neq 0$) and $\frac{a}{k}$ = integer = p (say)

$\therefore \quad \frac{a/k}{1/k} = \frac{p}{q} \quad \Rightarrow \quad a = \frac{p}{q} \quad \therefore \quad$ a = rational number

Ex.42 Given below is a partial graph of an even periodic function *f* whose period is 8. If [*] denotes greatest integer function then find the value of the expression.



$f(-3) + 2\,|f(-1)| + \left[f\left(\frac{7}{8}\right)\right] + f(0) + \text{arc}\cos(f(-2)) + f(-7) + f(20)$

Sol. f (–3) = f (3) = 2 [f (x) is an even function, $\therefore$ f (– x) = f (x)]

again f (–1) = f (1) = – 3

$\therefore$ 2 | f (–1) | = 2 | f (1) | = 2 | – 3 | = 6

from the graph, $-3 < f\left(\frac{7}{8}\right) < -2$ $\therefore$ $\left[f\left(\frac{7}{8}\right)\right] = -3$

f (0) = 0 (obviously from the graph)

$\cos^{-1}(f(-2)) = \cos^{-1}(f(2)) = \cos^{-1}(1) = 0$

f (–7) = f (– 7 + 8) = f (1) = – 3 [f (x) has period 8]

f (20) = f (4 + 16) = f (4) = 3 [f (nT + x) = f (x)]

sum = 2 + 6 – 3 + 0 + 0 – 3 + 3 = 5

Ex.43 Check whether the function defined by $f(x + \lambda) = 1 + \sqrt{2f(x) - f^2(x)}$ $\forall$ x $\in$ R, is periodic or not, if periodic, then find its period.

Sol. The given function is true if $2f(x) - f^2(x) \ge 0 \Rightarrow f(x)[f(x) - 2] \le 0 \Rightarrow 0 \le f(x) \le 2$ ....(i)

Also from the given function, it is clear that $f(x + \lambda) \ge 1 \Rightarrow$ $f(x) \ge 1$ ....(ii)

From (i) and (ii), we conclude that $1 \le f(x) \le 2$

Again, we have $\{f(x + \lambda) - 1\}^2 = 2f(x) - f^2(x) \Rightarrow \{f(x + \lambda) - 1\}^2 = -\{(f(x) - 1)^2\} + 1$ ...(iii)

Replacing x by x + λ in above equation, we get $\{f(x + 2\lambda) - 1\}^2 = -\{f(x + \lambda) - 1\}^2 + 1$ ...(iv)

From (iv) – (iii), we get

$\{f(x + 2\lambda) - 1\}^2 = \{f(x) - 1)\}^2 \Rightarrow f(x + 2\lambda) = f(x) \Rightarrow$ f is periodic function with period 2λ.

Ex.44 If the periodic function f(x) satisfies the equation $f(x + 1) + f(x - 1) = \sqrt{3}\, f(x)$ $\forall$ x $\in$ R then find the period of f(x)

Sol. We have $f(x + 1) + f(x - 1) = \sqrt{3}\, f(x)$ $\forall$ x $\in$ R ...(1)

Replacing x by x – 1 and x + 1 in (1) then $f(x) + f(x - 2) = \sqrt{3}\, f(x - 1)$ ...(2)

and $f(x + 2) + f(x) = \sqrt{3}\, f(x + 1)$ ...(3)

Adding (2) and (3), we get $2f(x) + f(x - 2) + f(x + 2) = \sqrt{3}\,(f(x - 1) + f(x + 1))$

$\Rightarrow$ $2f(x) + f(x - 2) + f(x + 2) = \sqrt{3} \cdot \sqrt{3}\, f(x)$ [From (1)]

$\therefore$ $f(x + 2) + f(x - 2) = f(x)$ ...(4)

Replacing x by x + 2 in equation (4) then f (x + 4) + f (x) = f (x + 2) ...(5)

Adding equations (4) and (5), we get f(x + 4) + f (x – 2) = 0 ...(6)

Again replacing x by x + 6 in (6) then f (x + 10) + f (x + 4) = 0 ...(7)

Subtracting (6) from (7), we get f (x + 10) – f (x – 2) = 0 ...(8)

Replacing x by x + 2 in (8) then f (x + 12) – f(x) = 0 or f (x + 12) = f(x)

Hence f(x) is periodic function with period 12.

Ex.45 Let f (x, y) be a periodic function satisfying f(x, y) = f(2x + 2y, 2y − 2x) for all x, y; Define g(x) = f(2^x , 0). Show that g(x) is a periodic function with period 12.

Sol. Since f(x, y) = f(2x + 2y, 2y − 2x) ...(1)

$\therefore$ f(2x + 2y, 2y − 2x)

= f(2(2x + 2y) + 2(2y − 2x), 2(2y − 2x) − 2(2x + 2y)) = f(8y, −8x) ...(2)

From (1) and (2) we get f(x, y) = f(8y, − 8x) ...(3)

$\therefore$ f(8y, − 8x) = f(8 (−8x), −8 (8y)) = f(−64, − 64y) ...(4)

From (3) and (4) we get f(x, y) = f(−64x, −64y) ...(5)

$\therefore$ f(−64x, − 64y) = f(−64 (−64x), −64(−64y)) = f(2^{12}x, 2^{12}y) ...(6)

From (5) and (6) we get f(x, y) = f(2^{12}x, 2^{12}y) $\Rightarrow$ f(x, 0) = f(2^{12}x, 0)

Replace x by 2^x the f(2^x, 0) = f(2^{x+12}, 0) $\Rightarrow$ g(x) = g(x + 12) {$\because$ g(x) = f(2^x, 0)}

Hence g(x) is periodic with period 12.

J. INVERSE OF A FUNCTION

Let $f: A \to B$ be a one–one & onto function, then their exists a unique function $g: B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1}: B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of inverse function :

(i) The inverse of a bijection is unique, and it is also a bijection.

(ii) If $f: A \to B$ is a bijection & $g: B \to A$ is the inverse of f, then fog = I_B and gof = I_A , where I_A & I_B are identity functions on the sets A & B respectively.

(iii) The graphs of f & g are the mirror images of each other in the line y = x.

(iv) Normally points of intersection of f and f^{-1} lie on the straight line y =x. However it must be noted that f(x) and $f^{-1}(x)$ may intersect otherwise also.

(v) In general fog(x) and gof(x) are not equal. But if either f and g are inverse of each other or atleast one of f, g is an identity function, then gof = fog.

(vi) If f & g are two bijections $f: A \to B$, $g: B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \circ g^{-1}$.

Ex.46 Find the inverse of the function f(x) = ln(x^2 + 3x +1); x $\in$ [1, 3] and assuming it to be an onto function.

Sol. Given f(x) = ln (x^2 + 3x + 1) $\therefore$ $f'(x) = \dfrac{2x+3}{(x^2+3x+1)} > 0\ \forall\ x \in [1, 3]$

which is a strictly increasing function. Thus f(x) is injective, given that f(x) is onto. Hence the given function f(x) is invertible. Now let y = f(x) = ln (x^2 + 3x + 1) then x = f^{-1} (y) ...(1)

and y = ln (x^2 + 3x + 1) $\Rightarrow$ $e^y = x^2 + 3x + 1$ $\Rightarrow$ $x^2 + 3x + 1 - e^y = 0$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9-4.1.(1-e^y)}}{2} = \frac{-3 \pm \sqrt{(5+4e^y)}}{2} = \frac{-3 + \sqrt{(5+4e^y)}}{2} \quad (\because x \in [1, 3]) \ ...(2)$$

From (1) and (2), we get $f^{-1}(y) = \dfrac{-3+\sqrt{(5+4e^y)}}{2}$ Hence $f^{-1}(x) = \dfrac{-3+\sqrt{(5+4e^x)}}{2}$

Ex.47 Find the inverse of the function $f(x) = \begin{cases} x, & x<1 \\ x^2, & 1\le x\le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Sol. Given $f(x) = f(x) = \begin{cases} x, & x<1 \\ x^2, & 1\le x\le 4 \\ 8\sqrt{x}, & x>4 \end{cases}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ....(1) $\therefore x = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1\le \sqrt{y}\le 4 \\ y^2/64, & y^2/64>4 \end{cases} = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1\le y\le 16 \\ y^2/64, & y>16 \end{cases}$

$f^{-1}(y) = \begin{cases} y, & y<1 \\ \sqrt{y}, & 1\le y\le 16 \\ y^2/64, & y>16 \end{cases}$ [From (1)]. Hence $f^{-1}(x) = \begin{cases} x, & x<1 \\ \sqrt{x}, & 1\le x\le 16 \\ y^2/64, & x>16 \end{cases}$.

Ex.48 A function $f : \left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$ defined as, $f(x) = x^2 - 3x + 4$. Then compute $f^{-1}(x)$ and find the solution of the equation, $f(x) = f^{-1}(x)$.

Sol. $f(x) = y = x^2 - 3x + 4 \quad \Rightarrow \quad x^2 - 3x + (4 - y) = 0$

$x = \frac{3 \pm \sqrt{9 - 4(4-y)}}{2} = \frac{3 + \sqrt{4y - 7}}{2} \qquad f^{-1}(y) = 3 + \frac{\sqrt{4x-7}}{2}$

graphs of $f^{-1}(x)$ & $f(x)$ intersect each other at $y = x \Rightarrow f(x) = x \Rightarrow x^2 - 3x + y = x \Rightarrow x = 2$

Ex.49 Let l_1 be the line $4x + 3y = 3$ and l_2 be the line $y = 8x$. L_1 is the line formed by reflecting l_1 across the line $y = x$ and L_2 is the line formed by reflecting l_2 across the x-axis. If θ is the acute angle between L_1 and L_2 such that $\tan\theta = a/b$, where a and b are coprime then find (a + b).

Sol. $l_1 : 4x + 3y = 3 \qquad f(x) = y = \frac{3-4x}{3}$ ....(1)

since f (x) and $f^{-1}(x)$ are the mirror images of each other in the line y = x hence we find $f^{-1}(x)$.

now $\quad y = f(x) \quad \Rightarrow \quad f^{-1}(y) = x$

from (1) $\quad x = \frac{3(1-y)}{4}$; $f^{-1}(y) = \frac{3(1-y)}{4} \quad \therefore \quad f^{-1}(x) = \frac{3(1-x)}{4}$

$4y = 3 - 3x \quad L_1 = 3x + 4y - 3 = 0$

$m_1 = -3/4 \quad$ |||ly $\quad L_2 = y = -8x$ with $m_2 = -8$

if θ is the acute angle between the lines $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{-8 + \frac{3}{4}}{1 + (-8)\left(-\frac{3}{4}\right)}\right| = \left|\frac{-29}{28}\right| \Rightarrow \frac{29}{28}$

$\Rightarrow \quad a = 29$ and $b = 28 \qquad \therefore \quad a + b = 29 + 28 = 57$